The Quantum Cosmological Wavefunction at Very Early Times for a Quadratic Gravity Theory

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Abstract. The quantum cosmological wavefunction for a quadratic gravity theory derived from the heterotic string effective action is obtained near the inflationary epoch and during the initial Planck era. Neglecting derivatives with respect to the scalar field, the wavefunction would satisfy a third-order differential equation near the inflationary epoch which has a solution that is singular in the scale factor limit $a(t) \to 0$. When scalar field derivatives are included, a sixth-order differential equation is obtained for the wavefunction and the solution by Mellin transform is regular in the $a \to 0$ limit. It follows that inclusion of the scalar field in the quadratic gravity action is necessary for consistency of the quantum cosmology of the theory at very early times.

The quadratic gravity theory derived from the four-dimensional heterotic string effective action obtained by orbifold compactification of the extra dimensions contains the coupling of the dilaton field to the Gauss-Bonnet term R^2_{GB} and a pseudo-scalar field to an $R\tilde{R}$ term [1]. While the quantum cosmology of this theory can be defined on a superspace which includes metrics with anisotropy, the properties of renormalizability in the generalized sense and unitarity occur only when the $R\tilde{R}$ term is set equal to zero. This can be achieved either by setting the pseudo-scalar field equal to zero without necessarily imposing any restrictions on the metric superspace or by confining the quantum cosmology to a minisuperspace of isotropic metrics. If the minisuperspace is chosen to be the space of Friedmann-Robertson-Walker metrics with scale factors $\{a(t)\}$ and scalar fields $\{\Phi(t)\}$, integration over the spatial coordinates gives a volume factor and leaves a one-dimensional action with an $\ddot{a}\dot{a}^2$ term. While a boundary term can be added to produce an action depending only on $a, \dot{a}$ and Φ, different quantization procedures may be used according to the choice of conjugate momentum variables. For an action with the second derivative of the scale factor, the conjugate momentum $P_{\dot{a}}$ can be introduced leading to a closed-form differential equation for the quantum cosmological wavefunction [2]. Use of the conventional momenta P_a, P_Φ gives rise to a Wheeler-De Witt equation which can be expanded in powers of $\frac{e^{-\Phi}}{g_4^2}$ with the higher-order terms representing corrections to the second-order Wheeler-DeWitt equation [3]. At very early times, however, the higher-order curvature terms will have a magnitude comparable to the Ricci scalar, so that a truncation of the series expansion in $\frac{e^{-\Phi}}{g_4^2}$ would no longer accurately define the Hamiltonian. Consequently, the sixth-order closed-form differential equation is preferable for the initial Planck era, whereas the series expansion of the Hamiltonian in $\frac{e^{-\Phi}}{g_4^2}$ would be valid in the inflationary epoch. While this procedure might be considered to be at variance with the standard canonical quantization of a gravitational field theory, different methods of quantization also have been found to be necessary to obtain the appropriate form for the Wheeler-DeWitt equation for higher-order gravity [4]. It also would be consistent with the freezing of the additional degree of freedom $P_{\dot{a}}$ during the transition from the Planck era to the inflationary epoch. For an inflationary model based on a slow-roll scalar potential, derivatives of the wavefunction with resepect to Φ can be neglected [5]. This approximation can be used to obtain a limiting form of the closed-form differential equation near the boundary between the Planck era and the inflationary epoch, which is a third-order differential equation in the scale factor a. After imposing matching conditions on the wavefunction and its first two derivatives at the boundary, the numerical solution extrapolated to $a = 0$ is found to be singular. An exact solution of the sixth-order differential equation by series expansion can be obtained by deducing the coefficients from a two-variable recursion relation. Techniques have recently been developed for solving two-variable recursion relations [6], but the approximate solution by Mellin transform is sufficient to establish regularity of the wavefunction in the $a \to 0$ limit.

1. Solution to Differential Equation for the Wavefunction near the Inflationary Epoch

Given an isotropic cosmology and specializing to Friedmann-Robertson-Walker metrics, the low-energy effective action for the heterotic string can be reduced to a one-dimensional integral

$$I \;=\; \int \; dt \; \left[(6a^2\ddot{a} + 6a\dot{a}^2 + 6aK) + \frac{1}{2}a^3\dot{\Phi}^2 + 6\frac{e^{-\Phi}}{g_4^2}\ddot{a}(\dot{a}^2+K)\right] \tag{1}$$

with conjugate momenta [2][3]

$$\begin{aligned} P_a &= \frac{\partial L}{\partial \dot{a}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{a}}\right) \;=\; 6\frac{e^{-\Phi}}{g_4^2}\dot{\Phi}(\dot{a}^2+K) \\ P_{\dot{a}} &= 6\left(a^2 + \frac{e^{-\Phi}}{g_4^2}(\dot{a}^2+K)\right) \\ P_\Phi &= \frac{\partial L}{\partial \dot{\Phi}} \;=\; a^3\dot{\Phi} \end{aligned} \tag{2}$$

so that the Hamiltonian is

$$\begin{aligned} H &= P_a\dot{a} + P_{\dot{a}}\ddot{a} + P_\Phi\dot{\Phi} - L \\ &= -6a(\dot{a}^2+K) + 6\frac{e^{-\Phi}}{g_4^2}\dot{\Phi}(\dot{a}^2+K)\dot{a} + \frac{1}{2}a^3\dot{\Phi}^2 \\ &= -g_4^2 P_\Phi^{-1}e^{\Phi}a^4P_a^4 \;+\; \frac{1}{2a^3}P_a^2 \;+\; \left[\frac{g_4^2}{6}P_\Phi^{-1}e^{\Phi}P_a^2a^3P_a \;-\; KP_a^2\right]^{\frac{1}{2}} \end{aligned} \tag{3}$$

and the Wheeler-DeWitt equation $H\Psi = 0$ can be transformed into

$$\begin{aligned} -\frac{g_4^2}{6}e^{-\Phi}\left(a^3\frac{\partial^4\Psi}{\partial a^3\partial\Phi} + 6a^2\frac{\partial^3\Psi}{\partial a^2\partial\Phi}\right) &+ Ke^{-2\Phi}\left(\frac{\partial^4\Psi}{\partial a^2\partial\Phi^2} - \frac{\partial^3\Psi}{\partial a^2\partial\Phi}\right) \\ = \; a^4g_4^4\left[4a^3\frac{\partial\Psi}{\partial a} + a^4\frac{\partial^2\Psi}{\partial a^2}\right] \;&+\; ag_4^2e^{-\Phi}\left(\frac{\partial^4\Psi}{\partial a\partial\Phi^3} + \frac{\partial^3\Psi}{\partial a\partial\Phi^2}\right) \\ +\; \frac{3}{2}g_4^2e^{-\Phi}\left(a\frac{\partial^2\Psi}{\partial a\partial\Phi} - \frac{\partial^3\Psi}{\partial\Phi^3}\right) \;&+\; \frac{1}{4a^6}e^{-\Phi}\frac{\partial}{\partial\Phi}\left(e^{-\Phi}\frac{\partial^5\Psi}{\partial\Phi^5}\right) \end{aligned} \tag{4}$$

The correspondence between the solutions of the sixth-order equation and the pseudo-differential equation $H\Psi = 0$ can be defined as follows. Given the two operators

$$\begin{aligned} \mathcal{A} &= g_4^2P_\Phi^{-1}e^{\Phi}a^4P_a \;-\; \frac{1}{2a^3}P_\Phi^2 \\ \mathcal{B} &= \left[\frac{g_4^2}{6}P_\Phi^{-1}e^{\Phi}P_a^2a^3P_a \;-\; KP_a^2\right]^{\frac{1}{2}} \end{aligned} \tag{5}$$

the solutions to $H\Psi = 0$ belong to $ker\ (\mathcal{A} - \mathcal{B})$. If $\mathcal{C} = e^{-\Phi} P_\Phi e^{-\Phi} P_\Phi (\mathcal{A}^2 - \mathcal{B}^2)$ then $ker\ \mathcal{C} = ker(\mathcal{A}^2 - \mathcal{B}^2) \cup (\mathcal{A}^2 - \mathcal{B}^2)^{-1}\ ker(e^{-\Phi} P_\Phi e^{-\Phi} P_\Phi)$. Since $(\mathcal{A}^2 - \mathcal{B}^2)\Psi = (\mathcal{A}+\mathcal{B})(\mathcal{A}-\mathcal{B})\Psi\ +\ (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})\Psi$,

$$
\begin{aligned}
&[ker(\mathcal{A}-\mathcal{B})\ \cup\ (\mathcal{A}-\mathcal{B})^{-1} ker(\mathcal{A}+\mathcal{B})]\ \cap\ \mathrm{ker}(\mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A})\ \subset\ ker(\mathcal{A}^2-\mathcal{B}^2) \\
&ker(\mathcal{A}-\mathcal{B}) = ker[(\mathcal{A}+\mathcal{B})(\mathcal{A}-\mathcal{B})] \cap (\mathcal{A}-\mathcal{B})^{-1}\ coker(\mathcal{A}+\mathcal{B}) \\
&\qquad \supset [ker(\mathcal{A}^2-\mathcal{B}^2)\ \cap\ \mathrm{ker}(\mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A})] \cap (\mathcal{A}-\mathcal{B})^{-1}\ coker(\mathcal{A}+\mathcal{B})
\end{aligned}
\tag{6}
$$

If $\Psi \in ker(\mathcal{A}-\mathcal{B})$, then $(\mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A})\Psi = (\mathcal{A}^2-\mathcal{B}^2)\Psi$, so that the equivalence $ker(\mathcal{A}-\mathcal{B}) = ker(\mathcal{A}^2-\mathcal{B}^2) \cap (\mathcal{A}-\mathcal{B})^{-1}\ coker(\mathcal{A}+\mathcal{B})$ is valid in this space. As $ker(\mathcal{A}^2-\mathcal{B}^2) = \mathcal{A}^{-1}\ ker(\mathcal{A}-\mathcal{B}) \cap \mathcal{B}^{-1}\ ker(\mathcal{A}-\mathcal{B})$, the restriction to the subdomain $ker(\mathcal{A}-\mathcal{B})$ implies that $ker(\mathcal{A}^2-\mathcal{B}^2)|_{ker(\mathcal{A}-\mathcal{B})} = \mathcal{A}^{-1}\ ker(\mathcal{A}-\mathcal{B})$ since $\mathcal{A}^{-1}\Psi = \mathcal{B}^{-1}\Psi$. Because

$$
ker(\mathcal{A}^2-\mathcal{B}^2) = ker\ \mathcal{C} \cap (\mathcal{A}^2-\mathcal{B}^2)^{-1} coker(e^{-\Phi} P_\Phi e^{-\Phi} P_\Phi)
\tag{7}
$$

the subdomain of $ker\ \mathcal{C}$ which satisfies $\mathcal{A}\Psi = \mathcal{B}\Psi$ is given by

$$
\left[ker\ \mathcal{C} \cap (\mathcal{A}^2-\mathcal{B}^2)^{-1} coker(e^{-\Phi} P_\Phi e^{-\Phi} P_\Phi)\right]\Big|_{ker(\mathcal{A}-\mathcal{B})} \cap (\mathcal{A}-\mathcal{B})^{-1}\ coker(\mathcal{A}+\mathcal{B}) \subset ker(\mathcal{A}-\mathcal{B})
\tag{8}
$$

However, the spaces of functions $coker(e^{-\Phi} P_\Phi e^{-\Phi} P_\Phi)$ and $coker(\mathcal{A}+\mathcal{B})$ are sufficiently large that the intersection does not exclude a significant portion of $ker\ \mathcal{C}|_{ker\ (\mathcal{A}-\mathcal{B})}$.

With the addition of the potential term $V(\Phi)$, the Hamiltonian becomes

$$
H\ =\ -g_4^2 P_\Phi^{-1} e^\Phi a^4 P_a\ +\ \frac{1}{2a^3} P_\Phi^2\ +\ \left[\frac{g_4^2}{6} P_\Phi^{-1} e^\Phi P_a^2 a^3 P_a - K P_a^2\right]^{\frac{1}{2}}\ +\ a^3 V(\Phi)
\tag{9}
$$

Transposing the square root expression leads to the equation

$$
\left(g_4^2 P_\Phi^{-1} e^\Phi a^4 P_a\ -\ \frac{1}{2a^3} P_\Phi^2\ -\ a^3 V(\Phi)\right)\Psi\ =\ \left[\frac{g_4^2}{6} P_\Phi^{-1} e^\Phi P_a^2 a^3 P_a - K P_a^2\right]^{\frac{1}{2}}\Psi
\tag{10}
$$

and upon squaring this equation, the new terms are

$$
\begin{aligned}
&\left(g_4^2 P_\Phi^{-1} e^\Phi a^4 P_a - \frac{1}{2a^3} P_\Phi^2\right)(-a^3 V(\Phi))\Psi \\
&\qquad +\ (-a^3 V(\Phi))\left(g_4^2 P_\Phi^{-1} e^\Phi a^4 P_a\ -\ \frac{1}{2a^3} P_\Phi^2\right)\Psi\ +\ a^6 V(\Phi)^2 \Psi
\end{aligned}
\tag{11}
$$

Multiplying by a single power of $e^{-\Phi} P_\Phi$ gives

$$
\begin{aligned}
&\left(g_4^2 a^4 P_a a^4 P_\Phi^{-1} e^\Phi\ -\ \frac{g_4^2}{2} P_\Phi^2 e^\Phi a P_a\ +\ \frac{1}{4a^6} e^{-\Phi} P_\Phi^5\ -\ \frac{g_4^2}{2} a^4 P_a \frac{1}{a^3} P_\Phi^2\right)\Psi \\
&\qquad =\ \frac{g_4^2}{6} P_a^2 a^3 P_a \Psi\ -\ K e^{-\Phi} P_\Phi P_a^2 \Psi
\end{aligned}
\tag{12}
$$

Since P_Φ is represented by the differential operator $-i\frac{\partial}{\partial\Phi}$ and its inverse P_Φ^{-1} by the integral $i\int\ d\Phi$. For a slow-roll potential $|V^{-1}(\Phi)V'(\Phi)| \ll 1$ and given that the Φ derivative of the potential and wave function is negligible in the semi-classical regime,

$$-ig_4^2 a^4 \frac{\partial}{\partial a} a^4 \frac{\partial}{\partial a} \int\ d\Phi\ e^\Phi \Psi + 2ig_4^2 a^7 V(\Phi)\frac{\partial \Psi}{\partial a} + 3ig_4^2 a^6 V(\Phi)\Psi = i\frac{g_4^2}{6}\frac{\partial^2}{\partial a^2}\left(a^3 \frac{\partial \Psi}{\partial a}\right) \tag{13}$$

giving rise to a third-order differential equation

$$\frac{a^2}{6}\frac{d^3\Psi}{da^3}\ +\ a(1+e^\Phi a^6)\frac{d^2\Psi}{da^2}\ +\ [(1+e^\Phi a^6) - 2a^6 V(\Phi)]\frac{d\Psi}{da}\ -\ 3a^5 V(\Phi)\Psi\ =\ 0 \tag{14}$$

The solution to this equation can be matched with the wave function satisfying the second Wheeler-DeWitt equation in the inflationary epoch after specification of the boundary conditions. This equation differs from a standard equation with form [7]

$$x^2 y''' + x(ax^n + b + c + 1)y'' + [\alpha x^{2n} + (ac + \beta) + \gamma + bc]y' + (c-1)(\alpha x^{2n} + \beta x^n + \gamma)\frac{y}{x} = 0 \tag{15}$$

by a single term, after setting $\gamma = 0$, $\alpha = 0$, $n = 6$, $a = 6e^\Phi$, $b + c + 1 = 6$, $bc = 6$, $ac + \beta = 12e^\Phi + \beta = 6e^\Phi - 12V(\Phi)$. It is known that the substitution $w = xy' + (c-1)y$ leads to a second-order equation of the form $x^2 w'' + x(ax^n + b)w' + (\alpha x^{2n} + \beta x^n + \gamma)y = 0$.

The $a \to 0$ limit of the differential equation (10) is

$$\frac{a^2}{6}\frac{d^3\Psi}{da^3} + a\frac{d^2\Psi}{da^2} + \frac{d\Psi}{da} = 0 \tag{16}$$

or equivalently

$$a^2 \frac{d^2\chi}{da^2} + a\frac{d\chi}{da} + 6\chi = 0 \tag{17}$$

with $\frac{d\Psi}{da}$ set equal to χ. If

$$H_0\ =\ a^2 \frac{d^2}{da^2}\ +\ 6a\frac{d}{da}\ +\ 6 \tag{18}$$

the full equation is

$$\begin{aligned} &(H_0 + H_1)\chi = 0 \\ &\qquad H_1\chi = 6a^7 e^\Phi \frac{d\chi}{da} + 6a^6(e^\Phi - 2V(\Phi))\chi - 18a^5 V(\Phi)\int da\chi \end{aligned} \tag{19}$$

Since the solution to $H_0\chi_0 = 0$ is $\chi_0 = \frac{A}{a^2} + \frac{B}{a^3}$,

$$\begin{aligned} H_1\chi_0 &= 6a^7 e^\Phi \left(-\frac{2A}{a^3} - \frac{3B}{a^4}\right) + 6a^6(e^\Phi - 2V(\Phi))\left(\frac{A}{a^2} + \frac{B}{a^3}\right) + 18a^5 V(\Phi)\left(\frac{A}{a} + \frac{B}{2a^2} - C\right) \\ &= 6Aa^4(V(\Phi) - e^\Phi) - 3B(V(\Phi) + 4e^\Phi) - 18Ca^5 V(\Phi) \end{aligned} \tag{20}$$

If $H_0\chi_1 = -H_1\chi_0$, then $H(\chi_0+\chi_1) = H_1\chi_1$. With the additional terms, a series solution $\sum_{n=0}^{\infty}\chi_n$ is obtained for χ with $H_0\chi_{n+1} = -H_1\chi_1$. Defining the Wronskian to be

$$W = \chi_{01}\frac{d}{da}\chi_{02} - \chi_{02}\frac{d}{da}\chi_{01} = \frac{A}{a^2}\frac{d}{da}\frac{B}{a^3} - \frac{B}{a^3}\frac{d}{da}\frac{A}{a^2} = -\frac{AB}{a^6} \tag{21}$$

and noting that the differential operator H_0 is $a^{-4}\frac{d}{da}a^6\frac{d}{da}+6$, the solution to the equation $H_0\chi_1 = -H_1\chi_0$ is

$$\chi_1 = k_1\chi_{01} + k_2\chi_{02} + \chi_{02}\int daa^{-2}\chi_{01}\frac{(-H_1\chi_0)}{W} - \chi_{01}\int daa^{-2}\chi_{02}\frac{(-H_1\chi_0)}{W}$$

which equals

$$\begin{aligned}\chi_1 = &\frac{A}{a^2}\int da\frac{B}{a^3}a^{-2}\frac{[6Aa^4(V(\Phi)-e^{\Phi}) - 3Ba^3(V(\Phi)+4e^{\Phi}) + 18Ca^5V(\Phi)]}{\frac{AB}{a^6}} \\ &- \frac{B}{a^3}\int da\frac{A}{a^2}a^{-2}\frac{[6Aa^4(V(\Phi)-e^{\Phi}) - 3Ba^3(V(\Phi)+4e^{\Phi}) + 18Ca^5V(\Phi)]}{\frac{AB}{a^6}} \\ = &\frac{B}{10}a^3(V(\Phi)+4e^{\Phi}) - \frac{A}{7}a^4(V(\Phi)-e^{\Phi}) + \frac{9C}{28}a^5V(\Phi)\end{aligned} \tag{23}$$

after setting $k_1 = k_2 = 0$, since an additional constant in the expression for each integral only leads to an overall shift in the value of A and B and therefore can be absorbed into those parameters. Then

$$\chi_0+\chi_1 = \frac{A}{a^2} + \frac{B}{a^3} + \frac{B}{10}a^3(V(\Phi)+4e^{\Phi}) - \frac{A}{7}a^4(V(\Phi)-e^{\Phi}) + \frac{9C}{28}a^5V(\Phi) \tag{24}$$

and

$$\Psi_0+\Psi_1 = -\frac{A}{a} - \frac{B}{2a^2} + D + \frac{B}{40}a^4(V(\Phi)+4e^{\Phi}) - \frac{A}{35}a^5(V(\Phi)-e^{\Phi}) - \frac{3C}{56}a^6V(\Phi) \tag{25}$$

Substitution of this function into the third-order differential equation gives $D = 0$, except at $a = 0$. Then

$$\Psi_0+\Psi_1 = -\frac{A}{a} - \frac{B}{2a^2} + \frac{B}{40}(V(\Phi)+4e^{\Phi})a^4 - \frac{A}{35}(V(\Phi)-e^{\Phi})a^5 - \frac{3C}{56}V(\Phi)a^6 \qquad a\neq 0 \tag{26}$$

At $a = 0$, it is possible to set $D = 1$ to satisfy the initial condition $\Psi(a=0) = 1$, although $D = 0$ for all a would be necessary for continuity of the wavefunction.

At the next order

$$
\begin{aligned}
&H_1\left[-\frac{a^4}{7}(V(\Phi)-e^\Phi)A+\frac{a^3}{10}(V(\Phi)+4e^\Phi)B+\frac{9C}{28}a^5V(\Phi)\right]\\
&\quad=6a^7e^\Phi\left[-\frac{4a^3}{7}(V(\Phi)-e^\Phi)A+\frac{3a^2}{10}(V(\Phi)+4e^\Phi)B+\frac{45C}{28}a^4V(\Phi)\right]\\
&\qquad+\ 6a^6(e^\Phi-2V(\Phi))\left[-\frac{a^4}{7}(V(\Phi)-e^\Phi)A+\frac{a^3}{10}(V(\Phi)+4e^\Phi)B+\frac{9C}{28}a^5V(\Phi)\right]\\
&\qquad-\ 18a^5V(\Phi)\left[-\frac{a^5}{35}(V(\Phi)-e^\Phi)A+\frac{a^4}{40}(V(\Phi)+4e^\Phi)B+\frac{3C}{56}a^6V(\Phi)\right]\\
&\quad=A\frac{-150e^\Phi+78V(\Phi)}{35}a^{10}(V(\Phi)-e^\Phi)+B\frac{48e^\Phi-33V(\Phi)}{20}a^9(V(\Phi)+4e^\Phi)\\
&\qquad+C\left(\frac{81}{7}e^\Phi-\frac{135}{28}V(\Phi)\right)a^{11}V(\Phi)
\end{aligned}
\tag{27}
$$

and the solution to the equation $H_0\chi_2=-H_1\chi_1$ is

$$
\begin{aligned}
\chi_2=&-\frac{1}{a^2}\int da\ a\Bigg[A\frac{-150e^\Phi+78V(\Phi)}{35}a^{10}(V(\Phi)-e^\Phi)+B\frac{48e^\Phi-33V(\Phi)}{20}a^9(V(\Phi)+4e^\Phi)\\
&\qquad+\ C\left(\frac{81}{7}e^\Phi-\frac{135}{28}V(\Phi)\right)a^{11}V(\Phi)\Bigg]\\
&+\frac{1}{a^3}\int da\ a^2\Bigg[A\frac{-150e^\Phi+78V(\Phi)}{35}a^{10}(V(\Phi)-e^\Phi)+B\frac{48e^\Phi-33V(\Phi)}{20}a^9(V(\Phi)+4e^\Phi)\\
&\qquad+\ C\left(\frac{81}{7}e^\Phi-\frac{135}{28}V(\Phi)\right)a^{11}V(\Phi)\Bigg]\\
=&\ A\frac{(75e^\Phi-39V(\Phi))(V(\Phi)-e^\Phi)a^{10}}{2730}-B\frac{(16e^\Phi-11V(\Phi))(V(\Phi)+4e^\Phi)a^9}{880}\\
&\qquad-C\frac{1}{182}\left(\frac{81}{7}e^\Phi-\frac{135}{28}\right)V(\Phi)a^{11}+...
\end{aligned}
\tag{28}
$$

It follows that

$$
\begin{aligned}
\chi=&\frac{A}{a^2}+\frac{B}{a^3}+\frac{B}{10}(V(\Phi)+4e^\Phi)a^3-\frac{A}{7}(V(\Phi)-e^\Phi)a^4+\frac{9C}{28}V(\Phi)a^5\\
&-\frac{B}{880}(16e^\Phi-11V(\Phi))(V(\Phi)+4e^\Phi)a^9+\frac{A}{2730}(75e^\Phi-39V(\Phi))(V(\Phi)-e^\Phi)a^{10}+\\
&\qquad-C\frac{1}{182}\left(\frac{81}{7}e^\Phi-\frac{135}{28}\right)V(\Phi)a^{11}+...
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
\Psi=&-\frac{A}{a}-\frac{B}{2a^2}+\frac{B}{40}(V(\Phi)+4e^\Phi)a^4-\frac{A}{35}(V(\Phi)-e^\Phi)a^5\\
&-\frac{B}{8800}(16e^\Phi-11V(\Phi))(V(\Phi)+4e^\Phi)a^{10}+\frac{A}{30030}(75e^\Phi-39V(\Phi))(V(\Phi)-e^\Phi)a^{11}\\
&-\frac{C}{728}\left(\frac{27}{7}e^\Phi-\frac{45}{28}\right)V(\Phi)a^{12}+...
\end{aligned}
\tag{30}
$$

This wavefunction may be extrapolated to $a = a_b$ representing the boundary between the initial era and the inflationary epoch. There it can be matched with the no-boundary wavefunction

$$\Psi_{0NB}(a) = \frac{Ai\left(K\left(\frac{36}{V}\right)^{\frac{2}{3}}\left(1 - \frac{a^2 V}{6K}\right)\right)}{Ai\left(K\left(\frac{36}{V}\right)^{\frac{2}{3}}\right)} \tag{31}$$

by equating the derivatives with respect to the scale factor up to second order.

A series solution of the form $\sum_n \ c_n \ (a - a_b)^n$, where a_b is the value of the scale factor at the boundary separating the initial Planck era from the inflationary epoch, can be used to match the solution to the third-order differential equation with the inflationary wave function. A recursion relation is obtained for the coefficients of this series

$$\begin{aligned}
&\frac{1}{6}\Big[(n+5)(n+6)(n+7)c_{n+7} + 2a_b(n+6)(n+7)(n+8)c_{n+8} \\
&\qquad\qquad + a_b^2(n+7)(n+8)(n+9)c_{n+9}\Big] \\
&+ \Big[(1 - 6a_b^5 e^{\Phi})(n+6)(n+7)c_{n+7} + (a_b - (1 + a_b^6 e^{\Phi}))(n+7)(n+8)c_{n+8} \\
&\quad - 15a_b^4 e^{\Phi}(n+5)(n+6)c_{n+6} - 20a_b^3(n+4)(n+5)c_{n+5}e^{\Phi} \\
&\quad - 15a_b^2 e^{\Phi}(n+3)(n+4)c_{n+4} - 6a_b(n+2)(n+3)c_{n+3}e^{\Phi} - (n+1)(n+2)c_{n+2}e^{\Phi}\Big] \\
&+ \Big[(1 + a_b^6 e^{\Phi} - 2a_b^6 V(\Phi))(n+7)c_{n+7} + 6a_b^5(e^{\Phi} - 2V(\Phi))(n+6)c_{n+6} \\
&\quad + 15a_b^4(e^{\Phi} - 2V(\Phi))(n+5)c_{n+5} + 20a_b^3(e^{\Phi} - 2V(\Phi))(n+4)c_{n+4} \\
&\quad + 15a_b^2(e^{\Phi} - 2V(\Phi))(n+3)c_{n+3} + 6a_b(e^{\Phi} - 2V(\Phi))(n+2)c_{n+2} \\
&\quad + (e^{\Phi} - 2V(\Phi))(n+1)c_{n+1}\Big] \\
&- 3\Big[c_{n+1} + 5a_b c_{n+2} + 10a_b^2 c_{n+3} + 10a_b^3 c_{n+4} + 5a_b^4 c_{n+5} + a_b^5 c_{n+6}\Big]V(\Phi) \ = \ 0
\end{aligned} \tag{32}$$

with c_0, c_1 and c_2 given by $\Psi_{inf.}(a_b)$, $\Psi'_{inf.}(a_b)$ and $\frac{1}{2}\Psi''_{inf.}(a_b)$. These values may be determined, for example, from the no-boundary wavefunction, whereas a_b, e^{Φ} and $V(\Phi)$ can be obtained from the time-dependence of the scale factor and the heterotic string potential.

The large-n limit of the recursion relation is $a_b^2 c_{n+9} + 2a_b c_{n+8} + c_{n+7} \ = \ 0$ with solution $c_n \ \sim \ (-a_b)^{-n} n$ and substitution of this functional dependence into the series gives $\sum_n \ (-a_b)^{-n} n \ (a - a_b)^n = \frac{1}{1 + a_b^{-1}(a - a_b)} = \frac{a_b}{a}$, which is the same divergent growth obtained earlier for the wave function as $a \to 0$. Given standard values for e^{Φ}, $V(\Phi)$ and a_b, the recursion relation can be solved numerically for the higher-order coefficients, and it may be confirmed that divergent behaviour is again obtained for the wavefunction near $a = 0$.

Divergence at $a = 0$ implies that a theory without dependence on the scalar field could not be physically consistent unless a lower bound is introduced for the scale factor. The existence of a lower bound would imply that the quantum theory must be defined on a minisuperspace $\{a(t), \Phi(t) \mid a(t) \geq a_0\}$ so that it describes perturbations about classical solutions with spacelike sections of non-zero minimal radius.

2. Solution to the Differential Equation for the Wavefunction in the Planck Era

The change of variables $w = e^{-\Phi}$ leads to the following form for the sixth-order differential equation:

$$\begin{aligned}
&\frac{g_4^2}{6} a^2 w^2 \left(a \frac{\partial^4 \Psi}{\partial a^3 \partial w} + 6 \frac{\partial^3 \Psi}{\partial a^2 \partial w} \right) \ + \ K w^3 \left(w \frac{\partial^4 \Psi}{\partial a^2 \partial w^2} + 2 \frac{\partial^3 \Psi}{\partial a^2 \partial w} \right) \\
&\quad = \ a^4 g_4^4 \left[4a^3 \frac{\partial \Psi}{\partial a} + a^4 \frac{\partial^2 \Psi}{\partial a^2} \right] \ - \ a g_4^2 w^3 \left(2 \frac{\partial^3 \Psi}{\partial a \partial w^2} + w \frac{\partial^4 \Psi}{\partial a \partial w^3} \right) \\
&\qquad - \frac{3}{2} g_4^2 w^2 \left(a \frac{\partial^2 \Psi}{\partial a \partial w} + \frac{\partial \Psi}{\partial w} + 3w \frac{\partial^2 \Psi}{\partial w^2} + w^2 \frac{\partial^3 \Psi}{\partial w^3} \right) \\
&\qquad + \frac{1}{4a^6} w^3 \left(2 \frac{\partial \Psi}{\partial w} + 46 w \frac{\partial^2 \Psi}{\partial w^2} + 115 w^2 \frac{\partial^3 \Psi}{\partial w^3} + 75 w^3 \frac{\partial^4 \Psi}{\partial w^4} + 16 w^4 \frac{\partial^5 \Psi}{\partial w^5} + w^5 \frac{\partial^6 \Psi}{\partial w^6} \right)
\end{aligned} \tag{33}$$

Given that the Mellin transform has the property $\mathcal{M}[f^{(n)}; s] = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} f^*(s-n)$, $\Psi^*(s, a)$ satisfies the equation

$$\begin{aligned}
&-a^2 \frac{g_4^2}{6} (s+1) \left[a \frac{d^3 \Psi^*(s+1, a)}{da} + 6 \frac{d^2 \Psi^*(s+1, a)}{da^2} \right] + K(s+1)(s+2) \frac{d^2 \Psi^*(s+2, a)}{da^2} \\
&\qquad = a^4 g_4^4 \left[4a^3 \frac{d\Psi^*(s, a)}{da} + a^4 \frac{d^2 \Psi^*(s, a)}{da^2} \right] + a g_4^2 (s+1)^2 (s+2) \frac{d\Psi^*(s+1, a)}{da} \\
&\qquad\quad + \frac{3}{2} g_4^2 \left[a(s+1) \frac{d\Psi^*(s+1, a)}{da} + (s+1)^3 \Psi^*(s+1, a) \right] \\
&\qquad\quad + \frac{1}{4a^6} [s^6 + 11s^5 + 50s^4 + 125s^3 + 205s^2 + 242s^s + 152] \Psi^*(s+2, a)
\end{aligned} \tag{34}$$

The difference-differential equation is

$$\begin{aligned}
&L_1(s, a) E_1^2 \Psi^*(s, a) + L_2(s, a) E_1 \Psi^*(s, a) + L_3(s, a) \Psi^*(s, a) = 0 \\
&L_1(s, a) = -K(s+1)(s+2) D_2^2 + \frac{1}{4a^6} [s^6 + 11s^5 + 50s^4 + 125s^3 + 205s^2 + 242s + 152] \\
&L_2(s, a) = a^2 \frac{g_4^2}{6} (s+1)(a D_2^3 + 6 D_2^2) + \frac{3}{2} g_4^2 (a(s+1) D_2 + (s+1)^3) \\
&\qquad\qquad + a g_4^2 (s+1)^2 (s+2) D_2 \\
&L_3(s, a) = a^4 g_4^4 [4a^3 D_2 + a^4 D_2^2] \\
&\qquad E_1 \Psi^*(s, a) = \Psi^*(s+1, a) \qquad\qquad D_2 \Psi^*(s, a) = \frac{d}{da} \Psi^*(s, a)
\end{aligned} \tag{35}$$

which has the asymptotic form

$$\frac{1}{4a^6} s^6 u_{s+2} + a g_4^2 (s+1)^2 (s+2) D_2 u_{s+1} + \frac{3}{2} g_4^2 (s+1)^2 u_{s+1} \simeq 0 \tag{36}$$

with solution

$$\begin{aligned} \frac{u_{s+2}}{u_{s+1}} &\simeq -\frac{(4a^6)}{s^3} \left[a g_4^2 D_2 + \frac{3}{2} g_4^2 \right] \\ u_{s+2} &\simeq (-1)^s \frac{(4a^6)^s}{\Gamma(s)^3} \left[a g_4^2 D_2 + \frac{3}{2} g_4^2 \right]^s \; u_2 \end{aligned} \tag{37}$$

The equation containing u_2 is

$$\begin{aligned} -K D_2^2 u_2 + \frac{1}{4a^6} 152 u_2 + a \frac{g_4^2}{6} (a D_2^3 + 6 D_2^2) u_1 + \frac{3}{2} g_4^2 \left(\frac{5}{3} a D_2 + 1 \right) u_1 \\ + a^4 g_4^4 (4a^3 D_2 + a^4 D_2^2) u_0 = 0 \end{aligned} \tag{38}$$

so that

$$u_2 = \left[K D_2^2 - \frac{38}{a^6} \right]^{-1} \left\{ a \frac{g_4^2}{6} (a D_2^3 + 6 D_2^2) u_1 + \frac{3}{2} g_4^2 \left(\frac{5}{3} a D_2 + 1 \right) u_1 + a^4 g_4^4 (4a^3 D_2 + a^4 D_2^2) u_0 \right\} \tag{39}$$

For large s, the Mellin transform of the wavefunction is

$$\begin{aligned} \Psi^*(s,a) \approx (-1)^s \frac{(4a^6)^{s-2}}{\Gamma(s-2)^3} \left[a g_4^2 D_2 + \frac{3}{2} g_4^2 \right]^{s-2} \left[K D_2^2 - \frac{38}{a^6} \right]^{-1} \left\{ a \frac{g_4^2}{6} (a D_2^3 + 6 D_2^2) \Psi^*(1,a) \right. \\ \left. + \frac{3}{2} \left(\frac{5}{3} a D_2 + 1 \right) \Psi^*(1,a) + a^4 g_4^4 (4a^3 D_2 + a^4 D_2^2) \Psi^*(0,a) \right\} \end{aligned} \tag{40}$$

Application of the inverse transform yields the wavefunction

$$\begin{aligned} \Psi(w,a) \approx \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} (-1)^s \frac{(4a^6)^{s-2}}{\Gamma(s-2)^3} \left[a g_4^2 D_2 + \frac{3}{2} g_4^2 \right]^{s-2} w^{-s} ds \cdot \left[K D_2^2 - \frac{38}{a^6} \right]^{-1} \\ \left\{ a \frac{g_4^2}{6} (a D_2^3 + 6 D_2^2) \Psi^*(1,a) + \frac{3}{2} g_4^2 \left(\frac{5}{3} a D_2 + 1 \right) \Psi^*(1,a) \right. \\ \left. + a^4 g_4^4 (4a^3 D_2 + a^4 D_2^2) \Psi^*(0,a) \right\} \end{aligned} \tag{41}$$

After shifting the variable s to $s-2$ and choosing the new contour to be the imaginary axis, the integral has the form

$$\frac{w^{-2}}{2\pi} \int_{-\infty}^{\infty} \Gamma(-iy)^3 \; (-4a^6)^{iy} \left[\frac{y \; sinh \; \pi y}{\pi} \right]^3 \left[a g_4^2 D_2 + \frac{3}{2} g_4^2 \right]^{iy} \; w^{-iy} \; dy \tag{42}$$

or equivalently

$$\begin{aligned}
&\frac{w^{-2}}{16\pi^4}\int_{-\infty}^{\infty}\Gamma(-iy)^3\ y^3\ e^{\left[3\pi+i\left[ln(-4a^6)+ln(ag_4^2D_2+\frac{3}{2}g_4^2)-ln\ w)\right]\right]y}\ dy\\
&\quad-\frac{3w^{-2}}{16\pi^4}\int_{-\infty}^{\infty}\Gamma(-iy)^3\ y^3\ e^{\left[\pi+i\left[ln(-4a^6)+ln(ag_4^2D_2+\frac{3}{2}g_4^2)-ln\ w)\right]\right]y}\ dy\\
&\quad+\frac{3w^{-2}}{16\pi^4}\int_{-\infty}^{\infty}\Gamma(-iy)^3\ y^3\ e^{\left[-\pi+i\left[ln(-4a^6)+ln(ag_4^2D_2+\frac{3}{2}g_4^2)-ln\ w)\right]\right]y}\ dy\\
&\quad-\frac{w^{-2}}{16\pi^4}\int_{-\infty}^{\infty}\Gamma(-iy)^3\ y^3\ e^{\left[-3\pi+i\left[ln(-4a^6)+ln(ag_4^2D_2+\frac{3}{2}g_4^2)-ln\ w)\right]\right]y}\ dy
\end{aligned}\tag{43}$$

The function

$$\frac{1}{\Gamma(s)}=se^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}\tag{44}$$

has no poles or essential singularities in the finite plane, and it decays at a factorial rate if $Re\ s>0$. Consider the contour obtained by rotating the upper and lower halves of the imaginary axis towards the real axis. Since the integrand vanishes on the circular contours C_1 and C_2, the integral over the imaginary axis equals the integral over a contour which traverses the positive real axis in opposite directions and therefore equals zero.

Given that equation (40) defines a close approximation to $\Psi^*(s,a)$ for $|s|\geq N_0$, a more

accurate formula for the wavefunction is

$$
\begin{aligned}
\Psi(w,a) \simeq \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} & (-1)^s \frac{(4a^6)^{s-2}}{\Gamma(s-2)^3}\left[a g_4^2 D_2 + \frac{3}{2} g_4^2\right]^{s-2} w^{-s} ds \cdot \left[K D_2^2 - \frac{38}{a^6}\right]^{-1} \\
& \left\{ a\frac{g_4^2}{6}(a D_2^3 + 6 D_2^2)\Psi^*(1,a) + \frac{3}{2} g_4^2 \left(\frac{5}{3} a D_2 + 1\right)\Psi^*(1,a) \right. \\
& \left. + a^4 g_4^4 (4a^3 D_2 + a^4 D_2^2)\Psi^*(0,a) \right\} \\
+ \frac{1}{2\pi i}\int_{-iN_0}^{iN_0} & \Psi^*(s,a) w^{-s} ds \\
- \frac{1}{2\pi i}\int_{-iN_0}^{iN_0} & (-1)^s \frac{(4a^6)^{s-2}}{\Gamma(s-2)^3}\left[a g_4^2 D_2 + \frac{3}{2} g_4^2\right]^{s-2} w^{-s} ds \cdot \left[K D_2^2 - \frac{38}{a^6}\right]^{-1} \\
& \left\{ a\frac{g_4^2}{6}(a D_2^3 + 6 D_2^2)\Psi^*(1,a) + \frac{3}{2} g_4^2 \left(\frac{5}{3} a D_2 + 1\right)\Psi^*(1,a) \right. \\
& \left. + a^4 g_4^4 (4a^3 D_2 + a^4 D_2^2)\Psi^*(0,a) \right\} \\
= \frac{1}{2\pi i}\int_{-iN_0}^{iN_0} & \Psi^*(s,a) w^{-s} ds \\
- \frac{1}{2\pi i}\int_{-iN_0}^{iN_0} & (-1)^s \frac{(4a^6)^{s-2}}{\Gamma(s-2)^3}\left[a g_4^2 D_2 + \frac{3}{2} g_4^2\right]^{s-2} w^{-s} ds \cdot \left[K D_2^2 - \frac{38}{a^6}\right]^{-1} \\
& \left\{ a\frac{g_4^2}{6}(a D_2^3 + 6 D_2^2)\Psi^*(1,a) + \frac{3}{2} g_4^2 \left(\frac{5}{3} a D_2 + 1\right)\Psi^*(1,a) \right. \\
& \left. + a^4 g_4^4 (4a^3 D_2 + a^4 D_2^2)\Psi^*(0,a) \right\}
\end{aligned}
\tag{45}
$$

because the first integral vanishes. The values of $\Psi^*(s,a)$ on the imaginary axis have not been given, but they can be defined by analytic continuation from the values on the real axis, which may be deduced from $\{\Psi^*(s,a) | -2 < s < 2\}$ through the recursion relation. The boundary conditions for the wavefunction can be transformed to conditions on the Mellin transform.

$$
\begin{aligned}
\Psi^*(s,a_b) &= \int_0^\infty w^{s-1}\ \Psi(w,a_b) = \int_0^\infty w^{s-1}\Psi_{inf.}(w,a_b) dw = \Psi^*(s,a_b) \\
\frac{\partial \Psi^*}{\partial a}(s,a_b) &= \frac{\partial \Psi^*_{inf.}}{\partial a}(s,a_b) \\
\frac{\partial^2 \Psi^*}{\partial a^2}(s,a_b) &= \frac{\partial^2 \Psi^*_{inf.}}{\partial a^2}(s,a_b)
\end{aligned}
\tag{46}
$$

Given $\Psi^*_{inf.}(s,a_b)$ for arbitrary values of s, the the higher derivatives $\Psi^*(s,a_b),\ -2 < s < 2$, can be obtained from the difference-differential equation (34) and $\Psi^*(s,a),\ -2 < s < 2$, can be extrapolated to $a = 0$.

The $a \to 0$ limit of the integral (41) is given by

$$
\begin{aligned}
&lim_{a\to 0}\frac{w^{-2}}{2\pi^4}\int_{-\infty}^{\infty}\Gamma(-iy)^3(-4a^6)^{iy}y^3\ sinh^3\pi y\ \left[ag_4^2D_2+\frac{3}{2}g_4^2\right]^{iy}\ w^{-iy}dy\\
&\qquad\cdot\left[KD_2^2-\frac{38}{a^6}\right]^{-1}\ \left\{a\frac{g_4^2}{6}(aD_2^3+6D_2^2)\Psi^*(1,a)+\frac{3}{2}g_4^2\left(\frac{5}{3}aD_2+1\right)\Psi^*(1,a)\right.\\
&\qquad\qquad\left.+a^4g_4^4(4a^3D_2+a^4D_2^2)\Psi^*(0,a)\right\}\\
&\propto\ \frac{w^{-2}}{2\pi^4}\int_{-\infty}^{\infty}\Gamma(-iy)^3(-4)^{iy}\delta(y)y^3sinh^3\pi y\ \left[\frac{3}{2}g_4^2\right]^{iy}\ w^{-iy}dy\\
&\qquad\cdot lim_{a\to 0}\frac{a^6}{K}D_2^{-2}\ \left\{a\frac{g_4^2}{6}(aD_2^3+6D_2^2)\Psi^*(1,a)+\frac{3}{2}g_4^2\left(\frac{5}{3}aD_2+1\right)\Psi^*(1,a)\right.\\
&\qquad\qquad\left.+a^4g_4^4(4a^3D_2+a^4D_2^2)\Psi^*(0,a)\right\}
\end{aligned}
\tag{47}
$$

and again the integral over y vanishes. Then

$$
\begin{aligned}
lim_{a\to 0}\Psi(w,a)\ &=\ \frac{1}{2\pi i}lim_{a\to 0}\int_{-iN_0}^{iN_0}\Psi^*(s,a)w^{-s}ds\\
&-\frac{1}{2\pi i}lim_{a\to 0}\int_{-iN_0}^{iN_0}(-1)^s\frac{(4a^6)^{s-2}}{\Gamma(s-2)^3}\left[ag_4^2D_2+\frac{3}{2}g_4^2\right]^{s-2}\ w^{-s}ds\\
&\qquad\left[KD_2^2-\frac{38}{a^6}\right]^{-1}\ \left\{g_4^2\left(\frac{a}{6}(aD_2^3+6D_2^2)+\frac{3}{2}\left(\frac{5}{3}aD_2+1\right)\right)\Psi^*(1,a)\right.\\
&\qquad\qquad\left.+a^4g_4^4(4a^3D_2+a^4D_2^2)\Psi^*(0,a)\right\}\\
&=\ \frac{1}{2\pi i}\ lim_{a\to 0}\ \int_{-iN_0}^{iN_0}\ \Psi^*(s,a)w^{-s}ds
\end{aligned}
\tag{48}
$$

because the second integral vanishes by the regularity of the integrand in the $a \to 0$ limit as any divergences in $\Psi^*(0,a)$ and $\Psi^*(1,a)$ could be absorbed by the factor of a^6 multiplying the operator D_2^{-2}. This implies that the quantum theory would be consistent in the $a \to 0$ limit when the scalar field degree of freedom is included.

Since the new terms in the sixth-order differential equation are given by

$$
\begin{aligned}
&e^{-\Phi}P_\Phi e^{-\Phi}P_\Phi\left(g_4^2P_\Phi^{-1}e^\Phi a^4P_a-\frac{1}{2a^3}P_\Phi^2\right)(-a^3V(\Phi))\Psi\\
&+\ e^{-\Phi}P_\Phi e^{-\Phi}P_\Phi(-a^3V(\Phi))\left(g_4^2P_\Phi^{-1}e^\Phi a^4P_a\ -\frac{1}{2a^3}P_\Phi^2\right)\Psi\ +\ e^{-\Phi}P_\Phi e^{-\Phi}P_\Phi a^6V(\Phi)^2\Psi
\end{aligned}
\tag{49}
$$

regularity of the integrand in the expression for $\Psi(w,a)$ and therefore consistency of the quantum cosmology of the theory in the $a \to 0$ limit will be maintained in the presence of a potential V(Φ).

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References

[1] I. Antoniadis, J. Rizos, and K. Tamvakis, Nucl. Phys. *B*415 (1994) 497-514

[2] S. Davis, Gen. Rel. Grav. 32(3) (2000) 541-551

[3] S. Davis and H. C. Luckock, Phys. Lett. 459*B* (2000) 408-421

[4] H.-J. Schmidt, Phys. Rev. *D*49(12)(1994) 6354 -6366

[5] A. Vilenkin, Phys. Rev. *D*33(12)(1986) 3560-3569

[6] X. Z. Zhong and D. Y. Liu, Acta Math. Appl. Sinica 23 (2000) 497-506

[7] A. D. Polyanin and V. F. Zaitsev, *Handbook of exact solutions for ordinary differential equations* (Boca Raton: CRC Press, 1995)